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Two Variables Operation Continuity of Effect Algebras

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In this paper, we study two variables operation continuity of lattice effect algebra with respect to its order topology.

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1. INTRODUCTION

Let L be a set with two special elements 0, 1, $\perp$ be a subset of $L \times L$. We denote $a \perp b$ if $(a, b) \in \perp$. Also, let $\oplus : \perp \to L$ be a binary operation. If the following axioms hold:

- (i) (Commutative Law). If $a, b \in L$ and $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$.
- (ii) (Associative Law). If $a, b, c \in L$, $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$, $a \perp (b \oplus c)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
- (iii) (Orthocomplementation Law). For each $a \in L$ there exists a unique $b \in L$ such that $a \perp b$ and $a \oplus b = 1$.
- (iv) (Zero-Unit Law). If $a \in L$ and $1 \perp a$, then $a = 0$.

Then $(L, \perp, \oplus, 0, 1)$ is said to be an *effect algebras* (Foulis and Bennett, 1994).

Let $(L, \perp, \oplus, 0, 1)$ be an effect algebra. If $a, b \in L$ and $a \perp b$ we say that a and b be *orthogonal*. If $a \oplus b = 1$ we say that b is the *orthocomplement* of a, and write $b = a'$. It is clear that $1' = 0$, $(a')' = a$, $a \perp 0$ and $a \oplus 0 = a$ for all $a \in L$.

We also say that $a \leq b$ if there exists $c \in L$ such that $a \perp c$ and $a \oplus c = b$. We may prove that $\leq$ is a *partial order* of L and satisfies that $0 \leq a \leq 1$, $a \leq$

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$b \Leftrightarrow b' \le a'$ and $a \le b' \Leftrightarrow a \perp b$ for $a, b \in L$. If $a \le b$, the element $c \in L$ such that $c \perp a$ and $a \oplus c = b$ is unique, and satisfies the condition $c = (a \oplus b')'$. It will be denoted by $c = b \ominus a$. If $a \le b$ but $a \ne b$, we write $a < b$.

The above showed that each effect algebra $(L, \perp, \oplus, 0, 1)$ has two binary operations $\oplus$ and $\ominus$.

If the partial order $\le$ of $(L, \perp, \oplus, 0, 1)$ defined as above is a *lattice*, then $(L, \perp, \oplus, 0, 1)$ is said to be a *lattice effect algebra*.

2. ORDER TOPOLOGY OF EFFECT ALGEBRAS

A partial order set $(\Lambda, \preceq)$ is said to be a *directed set*, if for all $\alpha, \beta \in \Lambda$, there exists $\gamma \in \Lambda$ such that $\alpha \preceq \gamma,\ \beta \preceq \gamma$.

If $(\Lambda, \preceq)$ is a directed set and for each $\alpha \in \Lambda$, $a_\alpha \in (L, \perp, \oplus, 0, 1)$, then $\{a_\alpha\}_{\alpha \in \Lambda}$ is said to be a *net* of $(L, \perp, \oplus, 0, 1)$.

Let $\{a_\alpha\}_{\alpha \in \Lambda}$ be a net of $(L, \perp, \oplus, 0, 1)$. Then we write $a_\alpha \uparrow$, when $\alpha \preceq \beta$, $a_\alpha \le a_\beta$. Moreover, if a is the supremum of $\{a_\alpha : \alpha \in \Lambda\}$, i.e., $a = \vee\{a_\alpha : \alpha \in \Lambda\}$, then we write $a_\alpha \uparrow a$.

Similarly, we may write $a_\alpha \downarrow$ and $a_\alpha \downarrow a$.

If $\{u_\alpha\}_{\alpha \in \Lambda}, \{v_\alpha\}_{\alpha \in \Lambda}$ are two nets of $(L, \perp, \oplus, 0, 1)$, for $u \uparrow u_\alpha \le v_\alpha \downarrow v$ means that $u_\alpha \le v_\alpha$ for all $\alpha \in \Lambda$ and $u_\alpha \uparrow u$ and $v_\alpha \downarrow v$. We write $b \le u_\alpha \uparrow u$ if $b \le u_\alpha$ for all $\alpha \in \Lambda$ and $u_\alpha \uparrow u$.

We say a net $\{a_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$ is *order convergent* to a point a of L if there exists two nets $\{u_\alpha\}_{\alpha\in\Lambda}$ and $\{v_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$ such that

$$a \uparrow u_\alpha \le a_\alpha \le v_\alpha \downarrow a.$$

Let $\mathcal{F} = \{F : F = \emptyset$ or $F \subseteq L$ and satisfies that for each net $\{a_\alpha\}_{\alpha\in\Lambda}$ of F if $\{a_\alpha\}_{\alpha\in\Lambda}$ is order convergent to a, then $a \in F\}$.

It is easy to prove that $\emptyset,\ L \in \mathcal{F}$ and if $F_1, F_2, \ldots, F_n \in \mathcal{F}$, then $\cup_{i=1}^n F_i \in \mathcal{F}$; if $\{F_\mu\}_{\mu\in\Omega} \subseteq \mathcal{F}$, then $\cap_{\mu\in\Omega} F_\mu \in \mathcal{F}$. Thus, the family $\mathcal{F}$ of subsets of L define a *topology* τ_0^L on $(L, \perp, \oplus, 0, 1)$ such that $\mathcal{F}$ consists of all closed sets of this topology. The topology τ_0^L is called the *order topology* of $(L, \perp, \oplus, 0, 1)$ (Birkhoff, 1948).

We can prove that the order topology τ_0^L of $(L, \perp, \oplus, 0, 1)$ is the finest (strongest) topology on L such that for each net $\{a_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$, if $\{a_\alpha\}_{\alpha\in\Lambda}$ is order convergent to a, then $\{a_\alpha\}_{\alpha\in\Lambda}$ must be topology τ_0^L convergent to a. But the converse is not true.

Moreover, it follows from the definition of the order topology τ_0^L of $(L, \perp, \oplus, 0, 1)$ that the subset B of $(L, \perp, \oplus, 0, 1)$ is not a τ_0^L-closed subset iff there exists a net $\{a_\alpha\}_{\alpha\in\Lambda}$ of B such that $\{a_\alpha\}_{\alpha\in\Lambda}$ is order convergent to a, but $a \notin B$.

It is easy to prove that each a and b of $(L, \perp, \oplus, 0, 1)$, the closed interval $[a, b]$ is a τ_0^L-closed subset of $(L, \perp, \oplus, 0, 1)$.

But for open interval, the conclusion does not hold in general.

Example 1. Let $L = [0, 1] \times [0, 1]$, $(x_1, x_2), (y_1, y_2) \in L$ and $(x_1, x_2) \oplus (y_1, y_2)$ be defined iff $x_1 + y_1 \leq 1$, $x_2 + y_2 \leq 1$, and $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$. It is easy to prove that $(L, \perp, \oplus, (0, 0), (1, 1))$ is an effect algebra. The open interval $((0, 0), (0, 1))$ is not a τ_0^L-open subset of $(L, \perp, \oplus, (0, 0), (1, 1))$.

Let $(L, \perp, \oplus, 0, 1)$ be an effect algebra, $a \in (L, \perp, \oplus, 0, 1)$. We denote $N(a)$ the set of all element c of $(L, \perp, \oplus, 0, 1)$ such that c can not compare with a. It follows from $[0, a]$ and $[a, 1]$ are τ_0^L-closed subsets of $(L, \perp, \oplus, 0, 1)$ that $N(a)$ is a τ_0^L-open subset of $(L, \perp, \oplus, 0, 1)$.

3. ORDER CONVERGENT PROPERTIES

For the order convergent properties of nets in lattice effect algebras, Riecanova proved the following conclusions (Riecanova, 1999):

Let $(L, \perp, \oplus, 0, 1)$ be a lattice effect algebra. For elements of L we have

(i) $b' \geq a_\alpha \downarrow a$ implies that $a_\alpha \oplus b \downarrow a \oplus b$.
(ii) $b \leq a_\alpha \uparrow a$ implies that $a_\alpha \ominus b \uparrow a \ominus b$.
(iii) $b' \geq a_\alpha$ order convergent to a implies that $a_\alpha \oplus b$ order convergent to $a \oplus b$.
(iv) $b \leq a_\alpha$ order convergent to a implies that $a_\alpha \ominus b$ order convergent to $a \ominus b$.

Now, we general conclusions (i)–(iv) to two variable operation cases, at first, the following lemma is useful.

Lemma 1. (Riecanova, 1999). *Let $(L, \perp, \oplus, 0, 1)$ be a lattice effect algebra, a, $b \in (L, 0, 1, \oplus)$. Then we have*

(1) *A net $\{a_\alpha\}_{\alpha \in \Lambda}$ of $[a, b]$ is order convergent to c in $(L, \perp, \oplus, 0, 1)$ iff $c \in [a, b]$ and $\{a_\alpha\}_{\alpha \in \Lambda}$ is order convergent to c in $[a, b]$.*
(2) *Let $\tau_0^{[a,b]}$ be the order topology on the subposet $[a, b]$ of the poset $(L, \leq)$. Then $\tau_0^L \cap [a, b] = \tau_0^{[a,b]}$.*

By applying Lemma 1, we can prove the following lemma easily.

Lemma 2. *Let $M = (L, \perp, \oplus, 0, 1) \times (L, \perp, \oplus, 0, 1)$, $(x_1, x_2), (y_1, y_2) \in M$. If $(x_1, x_2) \perp (y_1, y_2)$ iff $x_1 \perp y_1$ and $x_2 \perp y_2$. Then $(M, \perp, \oplus, (0, 0), (1, 1))$ is an effect algebra if $(x_1, x_2) \oplus (y_1, y_2)$ is defined by $(x_1 \oplus y_1, x_2 \oplus y_2)$. Moreover, if $a, b, c, d \in (L, \perp, \oplus, 0, 1)$, then the order topology $\tau_0^{[a,b]\times[c,d]}$ of $[a, b] \times [c, d]$ and $\tau_0^M \cap ([a, b] \times [c, d])$ are same, and the product topology $\tau_0^{[a,b]} \times \tau_0^{[c,d]}$ of $([a, b], \tau_0^{[a,b]}) \times ([c, d], \tau_0^{[c,d]})$ are also same.*

Theorem 1. *Let* $(L, \perp, \oplus, 0, 1)$ *be a lattice effect algebra. For nets of* $(L, \perp, \oplus, 0, 1)$ *we have*

(1) *If for each* $\alpha \in \Lambda$, $b'_\alpha \geq a_\alpha$, *then* $a_\alpha \uparrow a$ *and* $b_\alpha \uparrow b$ *imply that* $a_\alpha \oplus b_\alpha \uparrow a \oplus b$.
(2) *If there exists* $c \in (L, \perp, \oplus, 0, 1)$ *such that for each* $\alpha \in \Lambda$, $b_\alpha \leq c'$, $a_\alpha \leq c$, *then* $a_\alpha \downarrow a$ *and* $b_\alpha \downarrow b$ *imply that* $a_\alpha \oplus b_\alpha \downarrow a \oplus b$.
(3) *If there exists* $c \in (L, \perp, \oplus, 0, 1)$ *such that for each* $\alpha \in \Lambda$, $b_\alpha \leq c'$, $a_\alpha \leq c$, *then* a_α *is order convergent to* a *and* b_α *is order convergent to* b *imply that* $a_\alpha \oplus b_\alpha$ *is order convergent to* $a \oplus b$.
(4) *If there exists* $c \in (L, \perp, \oplus, 0, 1)$ *such that for each* $\alpha \in \Lambda$, $a_\alpha \leq c \leq b_\alpha$, *then* a_α *is order convergent to* a *and* b_α *is order convergent to* b *imply that* $b_\alpha \ominus a_\alpha$ *is order convergent to* $b \ominus a$.

Proof: For simplicity, we only prove (1) and (4).

(1) Let $b'_\alpha \geq a_\alpha$, $a_\alpha \uparrow a$ and $b_\alpha \uparrow b$. At first, we show that $b' \geq a$. In fact, note that $b' \downarrow b'_\alpha \geq a_\alpha \uparrow a$, so each α_1 and $\alpha_2 \in \Lambda$, there exists $\beta \in \Lambda$ such that $\beta \geq \alpha_1$ and $\beta \geq \alpha_2$, thus, $a_{\alpha_1} \leq a_\beta \leq b'_\beta \leq b'_{\alpha_2}$. It follows from $b'_\alpha \downarrow b'$ that for each α_1, we have $a_{\alpha_1} \leq b'$, therefore, $b' \geq a$ is obvious.

Now, we prove that $a_\alpha \oplus b_\alpha \uparrow a \oplus b$. It is clear that $a_\alpha \oplus b_\alpha \leq a \oplus b$. If $c \in (L, \perp, \oplus, 0, 1)$ such that each $\alpha \in \Lambda$, $a_\alpha \oplus b_\alpha \leq c$. Thus, each $\alpha \in \Lambda$, $a_\alpha \leq c \ominus b_\alpha$. It follows again from $a_\alpha \uparrow a$ and $c \ominus b_\alpha \downarrow c \ominus b$ that $a \leq c \ominus b$. So $a \oplus b \leq c$. This showed that $a_\alpha \oplus b_\alpha \uparrow a \oplus b$.

(4) Let $c \in (L, \perp, \oplus, 0, 1)$ such that each $\alpha \in \Lambda$, $a_\alpha \leq c \leq b_\alpha$, and a_α be order convergent to a, b_α be order convergent to b. It follows from Lemma 1 that a_α is order convergent to a in $[0, c]$ and b_α is order convergent to b in $[c, 1]$. So, there exist u_α and v_α in $[0, c]$, p_α and q_α in $[c, 1]$ such that $a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a$, $b \uparrow p_\alpha \leq b_\alpha \leq q_\alpha \downarrow b$. It follows from (1) and (2) of above easily that $b \ominus a \uparrow p_\alpha \ominus v_\alpha \leq b_\alpha \ominus a_\alpha \leq q_\alpha \ominus u_\alpha \downarrow b \ominus a$. So $b_\alpha \ominus a_\alpha$ is order convergent to $b \ominus a$. This theorem is proved. □

4. TWO VARIABLES OPERATION CONTINUITY

For the order topology continuity of one variable operations, Riecanova (1999) proved the following conclusions:

Let $(L, \perp, \oplus, 0, 1)$ be a lattice effect algebra. Then a net $\{a_\alpha\}_{\alpha \in \Lambda}$ of $(L, \perp, \oplus, 0, 1)$ has

(vi) If $b' \geq a_\alpha$ for all $\alpha \in \Lambda$, and $\{a_\alpha\}_{\alpha \in \Lambda}$ converges to a with respect to the order topology τ_0^L, then $\{a_\alpha \oplus b\}$ converges to $a \oplus b$ with respect to the order topology τ_0^L.

(vii) If $b \leq a_\alpha$ for all $\alpha \in \Lambda$, and $\{a_\alpha\}$ converges to a with respect to the order topology τ_0^L, then $\{a_\alpha \ominus b\}$ converges to $a \ominus b$ with respect to the order topology τ_0^L.
(viii) If $b \geq a_\alpha$ for all $\alpha \in \Lambda$, and $\{a_\alpha\}$ converges to a with respect to the order topology τ_0^L, then $\{b \ominus a_\alpha\}$ converges to $b \ominus a$ with respect to the order topology τ_0^L.

In order to study two variable operation continuity, we need the following famous topology conclusion:

Lemma 3. *Let (X, T_1) and (Y, T_2) be two topological spaces and $f : (X, T_1) \to (Y, T_2)$. Then f is a continuous map iff for each closed subset A of (Y, T_2), the inverse image $f^{-1}(A)$ of A is a closed subset of (X, T_1).*

Our main results are

Theorem 2. *Let $(L, \perp, \oplus, 0, 1)$ be a lattice effect algebra. Then we have*

(1) *If there exists $c \in (L, \perp, \oplus, 0, 1)$ such that each $\alpha \in \Lambda$, $b_\alpha \leq c'$, $a_\alpha \leq c$. Then a_α is order topology τ_0^L convergent to a and b_α is order topology τ_0^L convergent to b imply that $a_\alpha \oplus b_\alpha$ is order topology τ_0^L convergent to $a \oplus b$.*
(2) *If there exists $c \in (L, \perp, \oplus, 0, 1)$ such that each $\alpha \in \Lambda$, $a_\alpha \leq c \leq b_\alpha$. Then a_α is order topology τ_0^L convergent to a and b_α is order topology τ_0^L convergent to b imply that $b_\alpha \ominus a_\alpha$ is order topology τ_0^L convergent to $b \ominus a$.*

Proof: For simplicity, we only prove (1).

First, we define map $f : [0, c] \times [0, c'] \to L$ by $f(x, y) = x \oplus y$. Now, we only need to show that f is a continuous map of topological space $([0, c], \tau_0^{[0,c]}) \times ([0, c'], \tau_0^{[0,c']})$ into (L, τ_0^L).

Let B be a closed subset of (L, τ_0^L). If $f^{-1}(B)$ is not a closed subset of $([0, c], \tau_0^{[0,c]}) \times ([0, c'], \tau_0^{[0,c']})$, it follows from Lemma 2 that $f^{-1}(B)$ is also not a closed subset of $([0, c] \times [0, c'], \tau_0^{[0,c]\times[0,c']})$, so there exists a net (x_α, y_α) of $f^{-1}(B)$ which is order convergent to $(x, y) \in [0, c] \times [0, c']$, but $(x, y) \notin f^{-}(B)$. It is clear that x_α is order convergent to x and y_α is order convergent to y. It follows from Theorem 1 (3) that $x_\alpha \oplus y_\alpha$ is order convergent to $x \oplus y$. Note that $x_\alpha \oplus y_\alpha \in B$ and B is a τ_0^L-closed subset of $(L, \perp, \oplus, 0, 1)$, so, $x \oplus y \in B$, thus, we have $(x, y) \in f^{-}(B)$. This is a contradiction and the theorem is proved. □

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